Lecture 28

Physics 404

We reviewed the problem of N non-interacting spin-1/2 particles on a 1D lattice in an external magnetic field $\vec{B}$. The energy of each spin is $U = -\vec{\mu} \cdot \vec{B}$, which has one of two values $-\mu B$, or $+\mu B$. We can calculate the thermal average magnetic moment starting with the partition function $Z = \sum_s e^{-\varepsilon_s/\tau} = 2cosh\left(\frac{\mu B}{\tau}\right)$, as $m = \frac{1}{Z}\sum_s m_s\, e^{-\varepsilon_s/\tau} = \mu\, tanh\left(\frac{\mu B}{\tau}\right)$. We can calculate the magnetization M (magnetic moment per unit volume) of the entire sample as $M = \frac{N}{V} m = n\mu\, tanh\left(\frac{\mu B}{\tau}\right)$. This model of non-interacting spins predicts a paramagnetic state in which the sample develops a non-zero magnetization in the presence of an external field B, but the magnetization will disappear if the external field is turned off. Can we create a material that is ferromagnetic – i.e. it develops a magnetization in the presence of an external field, but it retains that magnetization even after the external field is turned off? The answer is yes, if we now include interactions between the spins.

The crudest way to include interactions is to make another "mean field approximation." We assume that the spins now exert "peer pressure" and try to create a state in which all of the spins are aligned the same way. This "pressure" is manifested through an internal field that all of the other spins exert on a given spin in the lattice. We ignore the details of the spin orientations around the given spin and simply posit the existence of a "mean field" $B_{Neighbor}$ that is produced by the "bath" of all the neighboring spins on the spin of interest. We simply take $B_{Neighbor}$ to be proportional to the posited magnetization of the system: $B_{Neighbor} = \lambda M$, where λ is a fixed number between 0 and 1. This leads to a self-consistency condition for the mean field approximation based on the above expression for the magnetization in the absence of an external field: $M = n\mu\, tanh\left(\frac{\mu B_{Neighbor}}{\tau}\right) = n\mu\, tanh\left(\frac{\mu\lambda M}{\tau}\right)$. This is a transcendental equation for the magnetization as a function of temperature. Define $m \equiv M/n\mu$ and $t \equiv \tau/n\mu^2\lambda$ and the self-consistency equation becomes simply $m = tanh(m/t)$. This equation can be solved graphically or numerically. The numerical result for $m(t)$ is shown below:

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mag[t_] := m /. FindRoot[m == Tanh[m / t], {m, 1}]

mag[0.5]

0.957504

Plot[ mag[t], {t, 0, 1.2}]
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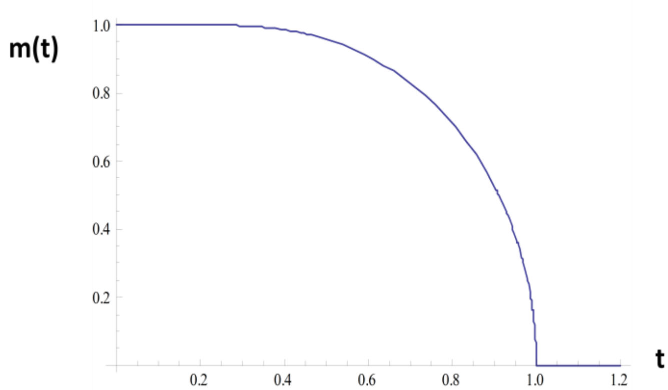

Note that $m(t > 1) = 0$, so there is only a "disordered" paramagnetic state at high temperatures. There is a non-zero magnetization only for $t < 1$, and this is the "ordered" ferromagnetic state. Here a large fraction of the spins have chosen to align in the same direction and produce a non-zero magnetization for the entire sample.

Note that the system has to choose one of two directions to align it's spins. This choice breaks the symmetry of the energy equation (which assigns the same energy to all the spins "up" and all the spins "down", in the absence of an external field). This is an example of spontaneous symmetry breaking, which is an important concept in physics. The non-zero magnetization is also an example of a "collective behavior" in which many individual microscopic spins work together to form a state with macroscopic properties (magnetization) that are easily measured.

We also discussed the Ising model, which is a more sophisticated treatment of the ferromagnetic to paramagnetic phase transition. This model can also account for anti-ferromagnetism, in which alternating spins point in opposite directions in the ordered state. The model is embodied in a Hamiltonian (total energy of all the spins): $H = \sum_{All\ sites\ i} -\vec{\mu_i} \circ \vec{B} + \sum_{\substack{Nearest \\ Neighbors\ \{i,j\}}} J_{ij}\, \vec{\mu_i} \circ \vec{\mu_j}$, where the spins $\vec{\mu_i}$ sit on a regular lattice in 1, 2 or 3 dimensions, and J_{ij} is called the "exchange interaction." The exchange interaction describes the interaction between neighboring spins and can be calculated from quantum mechanics. Positive values of J_{ij} encourage anti-ferromagnetism, while negative values favor ferromagnetic ordering. This model has been solved exactly in 1D and 2D, but not 3D. There are some very interesting applets that simulate the dynamics of this model available on the class web site.